A Recursive Algorithm for Generating the Equations of Motion of Spatial Mechanical Systems with Application to the Five-Point Suspension

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In this paper, a recursive formulation for generating the equations of motion of spatial mechanical systems is presented. The rigid bodies are replaced by a dynamically equivalent constrained system of particles which avoids introducing any rotational coordinates. For the open-chain system, the equations of motion are generated recursively along the serial chains using the concepts of linear and angular momenta. Closed-chain systems are transformed to open-chain systems by cutting suitable kinematic joints and introducing cut-joint constraints. The formulation is used to carry out the dynamic analysis of multi-link five-point suspension. The results of the simulation demonstrate the generality and simplicity of the proposed dynamic formulation.

Key Words : Multibody System Dynamics, Equations of Motion, System of Rigid Bodies, Mechanisms, Machine Theory

Nomenclature

$d_{i,j}$: Distance between points i and j

G_1 : Vector sum of the moments of the external forces and force couples acting on the body with respect to particle 1

$I_{\xi\xi}, I_{\eta\eta}, I_{\zeta\zeta}$: Moments of inertia of the body with respect to the body attached coordinate frame

$I_{\xi\eta}, I_{\xi\zeta}, I_{\zeta\eta}$: Products of inertia of the body with respect to the body attached coordinate frame

m : Mass of the body

m_i : Mass of particle i

$m_{i,j}$: Mass of the secondary particle that is located between the primary particles i and j

$\bar{\mathbf{r}}_G$: The position vector of the centre of mass of the body with respect to the body attached coordinate frame

$\bar{\mathbf{r}}_i$: Position vector of particle i with respect to the body attached coordinate frame

$\mathbf{r}_i$, $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}_i$: Position, velocity, and acceleration vectors of particle i with respect to an inertial reference frame

$\mathbf{r}_{i,j}$, $\dot{\mathbf{r}}_{i,j}$, $\ddot{\mathbf{r}}_{i,j}$: Relative position, velocity, and acceleration vectors between particles i and j

$\mathbf{r}_i^T\mathbf{r}_j$: Algebraic notation denotes the dot product operation $(\mathbf{r}_i, \mathbf{r}_j)$

$\tilde{\mathbf{r}}_i\mathbf{r}_j$: Algebraic notation denotes the cross product operation $(\mathbf{r}_i x \mathbf{r}_j)$

R : Vector sum of the external forces acting on the rigid body

ξ_i, η_i, ζ_i : Coordinates of particle i with respect to the body attached coordinate frame

1. Introduction

There are different formulations for the dynamic analysis of spatial mechanisms which depend on the system of coordinates used and in the way the kinematic constraint equations are introduced (Denavit and Hartenberg, 1955 ; Sheth and

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Uicker, 1972; Orlandea et al., 1977; Nikravesh, 1988). Each formulation has its own advantages and disadvantages depending on the application and the requirements. Some formulations are developed using a two-step transformation which leads to a system of equations of motion in matrix form. One method (Kim and Vanderploeg, 1986; Nikravesh and Gim, 1989) uses initially the absolute coordinate formulation where the location of each rigid body in the system is described in terms of a set of translational and rotational coordinates. However, this formulation has the disadvantage of the large number of coordinates defined. Then, the equations of motion are expressed in terms of the relative joint variables which determine the location of each body with respect to the adjacent body and they depend on the type of the kinematic joint connecting the two bodies. Another method uses initially the point coordinate formulation in which a dynamically equivalent constrained system of particles replaces the rigid bodies (Attia, 1993; Nikravesh and Affifi, 1994; Attia, 1998). The global motion of the constrained system of particles together with the constraints imposed upon them represent both the translational and rotational motions of the rigid body. The external forces and couples acting on the body are distributed over the system of particles. Due to the large number of differential-algebraic equations, the equations of motion which are expressed in terms of the Cartesian coordinates of the particles are rederived in terms of the relative joint variables. The main disadvantage of this two-step transformation is the necessity to transform at every time step from the joint variables to the original system which is computationally inefficient.

In this paper, a recursive formulation for generating the equations of motion of spatial mechanical systems is presented. The method is based upon the idea of replacing the rigid body by its dynamically equivalent constrained system of particles discussed in (Attia, 1993; Nikravesh and Affifi, 1994) with essential modifications and improvements. The concepts of the linear and angular momenta are used to formulate the rigid body dynamical equations. However, they are expressed in terms of the rectangular Cartesian coordinates of the equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal forces as in Newton-Euler formulation which results in a reduced system of differential-algebraic equations. Also, it expresses the general motion of the rigid body in terms of a set of Cartesian coordinates without introducing any rotational coordinates and the corresponding rotational transformation matrices and eliminates the necessity of distributing the external forces and couples over the particles. For the open-chain system, the equations of motion are generated recursively along the serial chains instead of the matrix formulation derived in (Attia, 1993; Nikravesh and Affifi, 1994). For the closed-chain system, the system is transformed to open-chain system by cutting suitable kinematic joints and introducing the cut-joint kinematic constraints. Then, the formulation is applied to study the dynamic analysis of the multi-link five-point motor-vehicle suspension. The results of the simulation demonstrate the generality and simplicity of the proposed method.

2. The Dynamic Formulation

2.1 Construction of the equivalent system of particles

The rigid body and its dynamically equivalent constrained system of particles should have the same mass, the same position of the centre of mass and the same inertia tensor with respect to a body attached coordinate frame. A system of ten particles is chosen to replace the rigid body as shown in Fig. 1. It should be pointed out that only four particles 1, ···, 4, which are denoted as primary particles, can dynamically replace the rigid body (Attia, 1993; Nikravesh and Affifi, 1994). However, additional six particles 5, ···, 10, which are denoted as the secondary particles, each is located at the midpoint between a pair of primary particles. The reason for introducing the secondary particles is to avoid the solution of non-linear algebraic equations due to the quadratic form of the second moment. The mass distributions to

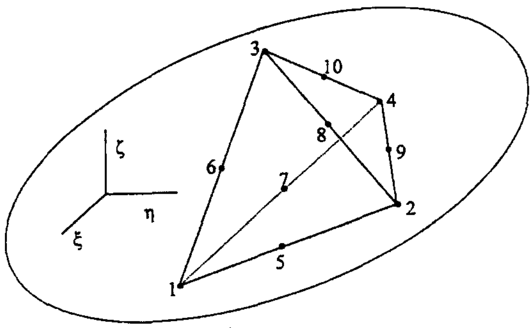


Fig. 1 The rigid body system wity an equivalent system of ten particles

points must satisfy the following conditions

$$m=\sum_{i=1}^{10} m_i \tag{1.1}$$

$$m\bar{\mathbf{r}}_G=\sum_{i=1}^{10} m_i\bar{\mathbf{r}}_i \tag{1.2}$$

$$I_{\xi\xi}=\sum_{i=1}^{10} m_i(\zeta_i^2+\eta_i^2) \tag{1.3}$$

$$I_{\eta\eta}=\sum_{i=1}^{10} m_i(\xi_i^2+\zeta_i^2) \tag{1.4}$$

$$I_{\zeta\zeta}=\sum_{i=1}^{10} m_i(\xi_i^2+\eta_i^2) \tag{1.5}$$

$$I_{\xi\eta}=\sum_{i=1}^{10} m_i\xi_i\eta_i \tag{1.6}$$

$$I_{\xi\zeta}=\sum_{i=1}^{10} m_i\xi_i\zeta_i \tag{1.7}$$

$$I_{\zeta\eta}=\sum_{i=1}^{10} m_i\zeta_i\eta_i \tag{1.8}$$

where m is the mass of the body, $\mathbf{r}_G$ is the position vector of the centre of mass of the body with respect to the body attached coordinate frame, $I_{\xi\xi}$, $I_{\eta\eta}$, $I_{\zeta\zeta}$ are the moments of inertia of the body with respect to the body attached coordinate frame, $I_{\xi\eta}$, $I_{\xi\zeta}$, $I_{\zeta\eta}$ are the products of inertia of the body with respect to the body attached coordinate frame, m_i is the mass of particle i, and $\bar{\mathbf{r}}_i$ is the position vector of particle i with respect to the attached coordinate frame. Equation (1) represents a 10×10 linear system of algebraic equations in 10 unknown masses of the primary and secondary particles.

2.2 Equations of motion of a single rigid body in spatial motion

For the equivalent system of particles, the force and moment equations take the form [10]

$$\mathrm{R}=\sum_{i=1}^{10} m_i\ddot{\mathbf{r}}_i \tag{2}$$

where R is the vector sum of the external forces acting on the rigid body and $\ddot{\mathbf{r}}_i$ is the acceleration vector of particle i. Also, the angular momentum equation for the whole system of particles with respect to particle 1 results in (Goldstein, 1950)

$$\mathrm{G}_1=\sum_{i=2}^{10} m_i\mathbf{r}_{i,1}x\ddot{\mathbf{r}}_i=\sum_{i=2}^{10} m_i\tilde{\mathbf{r}}_{i,1}\ddot{\mathbf{r}}_i \tag{3}$$

where G_1 is the vector sum of the moments of the external forces and force couples acting on the body with respect to particle 1 and $\mathbf{r}_{i,1}$ is the relative position vector between particles i and 1. The distance constraints between the ten particles are given as

$$\mathbf{r}_{2,1}^T\mathbf{r}_{2,1}-d_{2,1}^2=0,\ \mathbf{r}_{4,1}^T\mathbf{r}_{4,1}-d_{4,1}^2=0 \tag{4.1, 2}$$

$$\mathbf{r}_{4,2}^T\mathbf{r}_{4,2}-d_{4,2}^2=0,\ \mathbf{r}_{3,1}^T\mathbf{r}_{3,1}-d_{3,1}^2=0 \tag{4.3, 4}$$

$$\mathbf{r}_{3,2}^T\mathbf{r}_{3,2}-d_{3,1}^2=0,\ \mathbf{r}_{3,4}^T\mathbf{r}_{3,4}-d_{3,4}^2=0 \tag{4.5, 6}$$

$$\mathbf{r}_5-(\mathbf{r}_1+\mathbf{r}_2)/2=0,\ \mathbf{r}_6-(\mathbf{r}_1+\mathbf{r}_3)/2=0 \tag{4.7, 8}$$

$$\mathbf{r}_7-(\mathbf{r}_1+\mathbf{r}_4)/2=0,\ \mathbf{r}_8-(\mathbf{r}_2+\mathbf{r}_3)/2=0 \tag{4.9, 10}$$

$$\mathbf{r}_9-(\mathbf{r}_2+\mathbf{r}_4)/2=0,\ \mathbf{r}_{10}-(\mathbf{r}_3+\mathbf{r}_4)/2=0 \tag{4.11, 12}$$

The equations of motion (2), (3) and (4) represent a system of differential-algebraic equations that can be solved to determine the unknown acceleration vectors $\ddot{\mathbf{r}}_i$ of the particles at any instant of time. However, due to the large number of the geometric constraints the integration of these equations is inefficient. In the following section, some useful geometrical relationships are used to eliminate the majority of these constraints.

2.3 The reduced form of the equations of motion of a single rigid body

The reduced form of the equations of motion can be achieved in two steps. First, the secondary particles and their unknown accelerations can be easily eliminated by substituting the constraint Eqs. (4.7) to (4.12) into Eqs. (2) and (3) to obtain

$$\mathrm{R}=\sum_{i=1}^{4}\bar{m}_i\ddot{\mathbf{r}}_i \tag{5}$$

$$\mathrm{G}_1=\sum_{i=1}^{4}\mathrm{I}_i\ddot{\mathbf{r}}_i \tag{6}$$

where

$$\bar{m}_i=m_i+\sum_{\substack{j=1\\ j\neq i}}^{4}\frac{1}{2}\,m_{i,j} \tag{7}$$

$$\mathrm{A}_i=\bar{\bar{m}}_i\tilde{\mathbf{r}}_{i,1}+\sum_{\substack{j=2\\ j\neq i}}^{4}\frac{1}{4}\,m_{i,j}\tilde{\mathbf{r}}_{j,1} \tag{8}$$

$$\bar{\bar{m}}_i=m_i+\sum_{\substack{j=1\\ j\neq i}}^{4}\frac{1}{4}\,m_{i,j} \tag{9}$$

and where $m_{i,j}$ denotes the mass of the secondary particle that is located between the primary particles i and j ($m_{1,2}=m_5$, ⋯ etc.). Then, Eqs. (5) and (6) in addition to the remaining constraints Eqs. (4.1) to (4.6) represent the equations of motion for a single rigid body where only the primary particles stay. It should be pointed out that, in the case of a spatial rigid body with planar mass distribution, only three primary particles are replacing the rigid body and the equations of motion have identical form to Eqs. (5)-(9), but with upper limit of 3 instead of 4 for the summations. Similar treatment can be done for the rigid rod in spatial motion with two primary particles at both ends and one intermediate secondary particle.

A more reduced set of equations of motion can be derived by expressing the position vector of one of the primary particles in terms of the position vectors of the other three primary particles. We choose to express the coordinates of particle 3 in terms of the coordinates of particles 1, 2, and 4. As shown in Fig. 2, four invariant quantities $\bar{\lambda}$, $\bar{\nu}$, $\bar{\mu}$, and $\bar{\tau}$ can be estimated with the aid of the constraint Eqs. (4.4) to (4.6) that fix the distances between particle 3 and particles 1, 2, and 4 respectively. The invarient quantities take the form

$$\bar{\lambda}=\frac{|\,\mathbf{r}_{3,1}^{T}\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}\,|}{|\,\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}\,|} \tag{10.1}$$

$$\bar{\nu}=|\,\mathbf{r}_{5,1}\,| \tag{10.2}$$

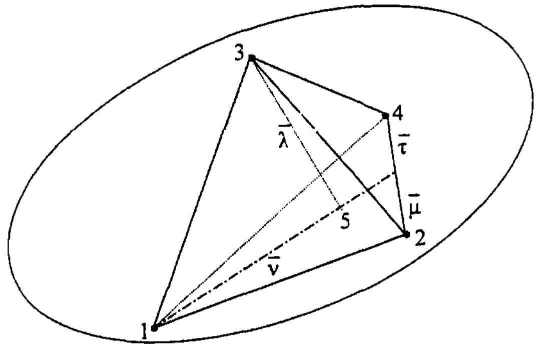


Fig. 2 The rigid body system with its equivalent primary particles indicating the invarient quantities

$$\bar{\mu}=\frac{|\,\mathbf{r}_{4,2}\,||\,\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{5,1}\,|}{|\,\tilde{\mathbf{r}}_{5,1}\mathbf{r}_{4,2}\,|} \tag{10.3}$$

$$\bar{\tau}=|\,\mathbf{r}_{4,2}\,|-\bar{\mu} \tag{10.4}$$

where

$$\mathbf{r}_{5,1}=\mathbf{r}_{3,1}-\bar{\lambda}\frac{\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}}{|\,\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}\,|}$$

Knowing the initial Cartesian coordinates of the primary particles, the invariant quantities are determined using Eqs. (10). In terms of these invariant quantities, the position vector of particle 3 is expressed as

$$\mathbf{r}_3=\mathbf{r}_1+\bar{\lambda}\frac{\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}}{|\,\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}\,|}+\bar{\nu}\frac{\bar{\mu}\mathbf{r}_{4,1}+\bar{\tau}\mathbf{r}_{2,1}}{|\,\bar{\mu}\mathbf{r}_{4,1}+\bar{\tau}\mathbf{r}_{2,1}\,|} \tag{11}$$

Since the quantities in the denominators in the right hand side of Eq. (11) are invarients we can rearrange the terms and obtain the simpler form

$$\mathbf{r}_3=\mathbf{r}_1+\lambda\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}+\mu\mathbf{r}_{4,1}+\tau\mathbf{r}_{2,1} \tag{12}$$

where

$$\lambda=\frac{\bar{\lambda}}{|\,\tilde{\mathbf{r}}_{2,1}\mathbf{r}_{4,1}\,|},\quad \mu=\frac{\bar{\nu}\bar{\mu}}{|\,\bar{\mu}\mathbf{r}_{4,1}+\bar{\tau}\mathbf{r}_{2,1}\,|}$$

$$\tau=\frac{\bar{\nu}\bar{\tau}}{|\,\bar{\mu}\mathbf{r}_{4,1}+\bar{\tau}\mathbf{r}_{2,1}\,|}$$

The corresponding velocity and acceleration vectors of particle 3 are estimated using the first and second time differentiations of Eq. (12) respectively which result in the following forms

$$\dot{\mathbf{r}}_3=\dot{\mathbf{r}}_1+\lambda(\tilde{\mathbf{r}}_{2,1}\dot{\mathbf{r}}_{4,1}+\dot{\tilde{\mathbf{r}}}_{2,2}\mathbf{r}_{4,1})+\mu\dot{\mathbf{r}}_{4,1}+\tau\dot{\mathbf{r}}_{2,1} \tag{13}$$

$$\begin{aligned}\ddot{\mathbf{r}}_3=\ddot{\mathbf{r}}_1&+\lambda(\tilde{\mathbf{r}}_{2,1}\ddot{\mathbf{r}}_{4,1}+\ddot{\tilde{\mathbf{r}}}_{2,1}\mathbf{r}_{4,1}+2\dot{\tilde{\mathbf{r}}}_{2,2}\dot{\mathbf{r}}_{4,1})\\&+\mu\ddot{\mathbf{r}}_{4,1}+\tau\ddot{\mathbf{r}}_{2,1}\end{aligned} \tag{14}$$

Equation (14) expresses the unknown acceleration vector of particle 3 in terms of the acceleration vectors of the other primary particles which eliminates the constraint Eqs. (4.4) to (4.7). Equation (14) can be put in the more convenient form

$$\begin{aligned}\ddot{\mathbf{r}}_3 &= (1-\mu-\tau+\lambda\tilde{\mathbf{r}}_{4,2})\ddot{\mathbf{r}}_1 \\ &\quad + (\tau-\lambda\tilde{\mathbf{r}}_{4,1})\ddot{\mathbf{r}}_2 + (\mu+\lambda\tilde{\mathbf{r}}_{2,1})\ddot{\mathbf{r}}_4\end{aligned} \tag{15}$$

Substituting the derived acceleration vector of particle 3 from Eq. (15) into Eqs. (5) and (6), then the differential equations of motion take the modified form

$$\begin{aligned}\mathbf{R} &= \{\bar{m}_1+\bar{m}_3(1-\mu-\tau+\lambda\tilde{\mathbf{r}}_{4,2})\}\ddot{\mathbf{r}}_1 \\ &\quad +\{\bar{m}_2+\bar{m}_3(\tau-\lambda\tilde{\mathbf{r}}_{4,1})\}\ddot{\mathbf{r}}_2 \\ &\quad +\{\bar{m}_4+\bar{m}_3(\mu+\lambda\tilde{\mathbf{r}}_{2,1})\}\ddot{\mathbf{r}}_4 + 2\lambda\bar{m}_3\dot{\tilde{\mathbf{r}}}_{2,1}\dot{\mathbf{r}}_{4,1}\end{aligned} \tag{16}$$

$$\begin{aligned}\mathrm{G}_1 &= \{\mathrm{A}_1+\mathrm{A}_3(1-\mu-\tau+\lambda\tilde{\mathbf{r}}_{4,2})\}\ddot{\mathbf{r}}_1 \\ &\quad +\{\mathrm{A}_2+\mathrm{A}_3(\tau-\lambda\tilde{\mathbf{r}}_{4,1})\}\ddot{\mathbf{r}}_2 \\ &\quad +\{\mathrm{A}_4+\mathrm{A}_3(\mu+\lambda\tilde{\mathbf{r}}_{2,1})\}\ddot{\mathbf{r}}_4 + 2\lambda\mathrm{A}_3\dot{\tilde{\mathbf{r}}}_{2,1}\dot{\mathbf{r}}_{4,1}\end{aligned} \tag{17}$$

Equations (16) and (17) in addition to the constraint Eqs. (4.1) to (4.3) represent the equations of motion of a single floating rigid body in spatial motion. It can be solved at every time step to determine the unknown acceleration components of particles 1, 2, and 4. Consequently, Eq. (15) can be used to determine the acceleration components of particle 3. The acceleration components of the particles are integrated numerically, knowing their Cartesian coordinates and velocities at a certain time step, to determine the positions and velocities for the next time step. The translational motion of the particles determines completely the translational and rotational motion of the rigid body. If the rigid body is rotating about a fixed point, then particle 1 may be located at the centre of this joint. In this case, Eq. (17) and Eqs. (4.1) to (4.3) are used to solve for the unknown Cartesian accelerations of particles 2 and 4. Equation (16) can be solved to determine the unknown reaction forces at the joint.

If the rigid body is rotating about a fixed axis, then particles 1 and 2 can be located along the axis of the joint to define its direction. The projection of the moments in Eq. (17) along the direction of the axis of the revolute joint in addition to the constraint Eqs. (4.2) and (4.3) can be used to determine the unknown acceleration vector of particle 4. Then, Eq. (16) may be used to get the reactions at the axis of the revolute joint.

2.4 Equations of motion of a serial chain of rigid bodies

Figure 3 shows a serial chain of N rigid bodies with the equivalent system of (3N+1) particles where connected particles are unified from both bodies. For the last body "N" in the chain, the equations of motion are derived in a similar way as Eq. (17) and Eqs. (4.1) to (4.3) of a single rigid body. The angular momentum equation takes the form

$$\begin{aligned}\mathrm{G}_{3N-2} &= \{\mathrm{A}_{3N-2}+\mathrm{A}_{3N}(1-\mu_N-\tau_N+\lambda_N\tilde{\mathbf{r}}_{3N+1,3N-1})\}\ddot{\mathbf{r}}_{3N-2} \\ &\quad +\{\mathrm{A}_{3N-1}+\mathrm{A}_{3N}(\tau_N-\lambda_N\tilde{\mathbf{r}}_{3N+1,3N-2})\}\ddot{\mathbf{r}}_{3N-1} \\ &\quad +\{\mathrm{A}_{3N+1}+\mathrm{A}_{3N}(\mu_N+\lambda_N\tilde{\mathbf{r}}_{3N-1,3N-2})\ddot{\mathbf{r}}_{3N+1} \\ &\quad +2\lambda_N\mathrm{A}_{3N}\dot{\tilde{\mathbf{r}}}_{3N-1,3N-2}\dot{\mathbf{r}}_{3N+1,3N-2}\end{aligned} \tag{18}$$

where

$$\mathrm{A}_{3N} = \bar{m}_{3N}\tilde{\mathbf{r}}_{3N,3N-2} + \sum_{\substack{i=3N-1 \\ i\neq 3N}}^{3N+1} \frac{1}{4} m_{3N,i}\tilde{\mathbf{r}}_{i,3N-2}$$

$$\bar{m}_{3N} = m_{3N} + \sum_{\substack{i=3N-2 \\ i\neq 3N}}^{3N+1} \frac{1}{4} m_{3N,i}$$

where G_{3N-2} is the sum of the moments of the external forces and force couples acting on body N with respect to the location of particle 3N−2. The acceleration equations of the distance constraint between primary particles belonging to body N are given as

$$\begin{aligned}&\mathbf{r}^T_{3N-2,3N-1}\ddot{\mathbf{r}}_{3N-2}+\mathbf{r}^T_{3N-1,3N-2}\ddot{\mathbf{r}}_{3N-1} \\ &= -\dot{\mathbf{r}}^T_{3N-1,3N-2}\dot{\mathbf{r}}_{3N-1,3N-2}\end{aligned} \tag{19.1}$$

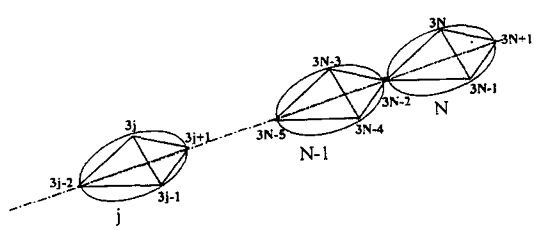


Fig. 3 Serial chain of N rigid bodies with an equivalent system of primary particles

$$\begin{aligned}&\mathbf{r}^T_{3N-2,3N+1}\ddot{\mathbf{r}}_{3N+2}+\mathbf{r}^T_{3N+1,3N-2}\ddot{\mathbf{r}}_{3N+1}\\&=-\dot{\mathbf{r}}^T_{3N+1,3N-2}\dot{\mathbf{r}}_{3N+1,3N-2}\end{aligned}\tag{19.2}$$

$$\begin{aligned}&\mathbf{r}^T_{3N-1,3N+1}\ddot{\mathbf{r}}_{3N-1}+\mathbf{r}^T_{3N+1,3N-1}\ddot{\mathbf{r}}_{3N+1}\\&=-\dot{\mathbf{r}}^T_{3N+1,3N-1}\dot{\mathbf{r}}_{3N+1,3N-1}\end{aligned}\tag{19.3}$$

Addition of one more body in the chain leads to the inclusion of an angular momentum vector equation that takes into consideration the contributions of all the ascending bodies in the chain together with three distance constraint equations between the particles belonging to this body. These six scalar equations are appended to the equations of motion derived for the leading bodies in the chain. For body j, the appended equations of motion take the form

$$\begin{aligned}\mathrm{G}_{3j-2}=\sum_{k=j}^{N}&\{\mathrm{A}_{3k-2}+\mathrm{A}_{3k}(1-\mu_k-\tau_k+\lambda_k\tilde{\mathbf{r}}_{3kk+1,3k-1})\}\ddot{\mathbf{r}}_{3k-2}\\&+\{\mathrm{A}_{3k-1}+\mathrm{A}_{3k}(\tau_k-\lambda_k\tilde{\mathbf{r}}_{3k+1,3k-2})\}\ddot{\mathbf{r}}_{3k-1}\\&+\{\mathrm{A}_{3k+1}+\mathrm{A}_{3k}(\mu_k+\lambda_k\tilde{\mathbf{r}}_{3k-1,3k-2})\}\ddot{\mathbf{r}}_{3k+1}\\&+2\lambda_k\mathrm{A}_{3k}\dot{\tilde{\mathbf{r}}}_{3k-1,3k-2}\dot{\mathbf{r}}_{3k+1,3k-2}\end{aligned}\tag{20}$$

where

$$\mathrm{A}_{3k}=\bar{m}_{3k}\tilde{\mathbf{r}}_{3k,3j-2}+\sum_{\substack{i=3k-1\\i\neq 3k}}^{3k+1}\frac{1}{4}m_{3k,i}\tilde{\mathbf{r}}_{i,3j-2}$$

$$\bar{m}_{3k}=m_{3k}+\sum_{\substack{i=3k-2\\i\neq 3k}}^{3k+1}\frac{1}{4}m_{3k,i}$$

$$\begin{aligned}&\mathbf{r}^T_{3j-2,3j-1}\ddot{\mathbf{r}}_{3j-2}+\mathbf{r}^T_{3j-1,3j-2}\ddot{\mathbf{r}}_{3j-1}\\&=-\dot{\mathbf{r}}^T_{3j-1,3j-2}\dot{\mathbf{r}}_{3j-1,3j-2}\end{aligned}\tag{21.1}$$

$$\begin{aligned}&\mathbf{r}^T_{3j-2,3j+1}\ddot{\mathbf{r}}_{3j-2}+\mathbf{r}^T_{3j+1,3j-2}\ddot{\mathbf{r}}_{3j+1}\\&=-\dot{\mathbf{r}}^T_{3j+1,3j-2}\dot{\mathbf{r}}_{3j+1,3j-2}\end{aligned}\tag{21.2}$$

$$\begin{aligned}&\mathbf{r}^T_{3j-1,3j+1}\ddot{\mathbf{r}}_{3j-1}+\mathbf{r}^T_{3j+1,3j-1}\ddot{\mathbf{r}}_{3j+1}\\&=-\dot{\mathbf{r}}^T_{3j+1,3j-1}\dot{\mathbf{r}}_{3j+1,3j-1}\end{aligned}\tag{21.3}$$

where

$$\begin{aligned}\ddot{\mathbf{r}}_{3j}=&(1-\mu-\tau+\lambda\tilde{\mathbf{r}}_{3j+1,3j-1})\ddot{\mathbf{r}}_{3j-2}\\&+(\tau-\lambda\tilde{\mathbf{r}}_{3j+1,3j-2})\ddot{\mathbf{r}}_{3j-1}+(\mu+\lambda\tilde{\mathbf{r}}_{3j-1,3j-2})\ddot{\mathbf{r}}_{3j+1}\end{aligned}$$

according to Eq. (15) has been used. If body "j" is the floating base body in the chain then, three linear momentum equations, similar to Eq. (16), are required to solve for the unknown acceleration components of particle 1. These linear momentum equations equate the sum of the external forces acting on all the bodies in the chain to the time rate of change of the vectors of linear momentum of all the equivalent particles that replace the chain which take the form

$$\begin{aligned}\mathrm{R}=\sum_{k=j}^{N}&\{\bar{m}_{3k-2}+\bar{m}_{3k}(1-\mu_k-\tau_k+\lambda_k\tilde{\mathbf{r}}_{3k+1,3k-1})\}\ddot{\mathbf{r}}_{3k-2}\\&+\{\bar{m}_{3k-1}+\bar{m}_{3k}(\tau_k-\lambda_k\tilde{\mathbf{r}}_{3k+1,3k-2})\}\ddot{\mathbf{r}}_{3k-1}\\&+\{\bar{m}_{3k+1}+\bar{m}_{3k}(\mu_k+\lambda_k\tilde{\mathbf{r}}_{3k-1,3k-2})\}\ddot{\mathbf{r}}_{3k+1}\\&+2\lambda_k\bar{m}_{3k}\dot{\tilde{\mathbf{r}}}_{3k-1,3k-2}\dot{\mathbf{r}}_{3k+1,3k-2}\end{aligned}\tag{22}$$

where

$$\bar{m}_{3k}=m_{3k}+\sum_{\substack{i=3k-2\\i=3k}}^{3k+1}\frac{1}{2}m_{3k,i}$$

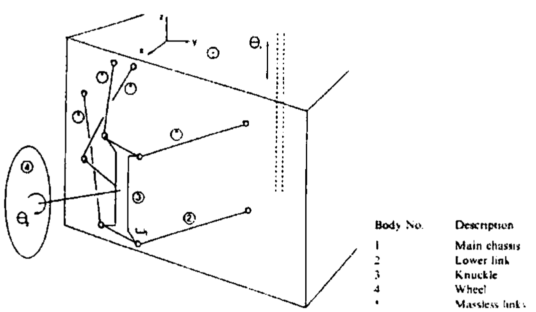
If body j is connected to body j-1 by a revolute joint, then we take the projection of all the moment vectors in Eq. (20) along the axis of the joint which is defined by two particles from both bodies that are commonly located on it. Two additional distance constraints, that fix the distances between the remaining fourth particle and the other two particles along the axis of the joint, together with the angular momentum equation can be used to solve for the acceleration vector of the fourth particle on body j.

In general, for a serial chain of N bodies, an equivalent system of (3N+1) primary particles and 6N secondary particles is first constructed. Then, by eliminating all the secondary particles and N primary particles, we are left with 2N+1 particles and consequently, 6N+3 unknown acceleration components. To solve for these unknowns, 3N angular momentum equations can be generated recursively along the chain together with 3N distance constraints between the particles located on each body, in case of all are spherical joints. In the case of a revolute joint, one angular momentum scalar equation and two distance constraints are used. Finally, three linear momentum equations can be used to solve for the unknown acceleration components of particle 1 if body 1 is floating or for the unknown reaction forces if there is a fixation at point 1.

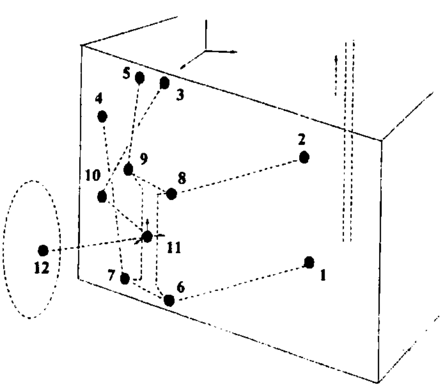
In the case of an open-chain system or closed-chain system, it can be transformed to a system of serial chains by cutting suitable joints and consequently cut-joint constraints are introduced. In the case of a closed-chain system, the cut-joints avoids the need to introduce loop closure equations and the corresponding loop closure constraint forces and then allows the use of the laws of momentum/moment of momentum with respect to a joint axis. Equivalent particles are conveniently chosen to locate at the positions of the connection joints and in terms of their Cartesian coordinates the cut-joint constraint equations are easily formulated. The cut-joints kinematic constraints substitute for the unknown cut-joints constraint reaction forces that appear explicitly in the linear and angular momentum equations generated recursively along the separated serial chains.

3. Dynamic Analysis of the Multi-Link Five-Point Suspension

Figure 4(a) presents a quarter car with five-point suspension system. This suspension is usually used for driven rear axles of current production of Mercedes-Benz cars, Mazda 929, some BMW and Toyota Supra cars. The mechanical system consists of a main chassis, a five-point suspension sub-system, a steering rod, and a wheel. A suspension spring and a shock absorber are included. The system constitutes four closed loops due to the four massless links connecting the chassis to the Knuckle. The chassis is constrained to move vertically which can be modelled as a translational joint with axis vertical. The wheel is analytically modelled in the radial direction by an equivalent linear translational spring system which also has damping characteristics. The system has three degrees of freedom. The chassis has one degree of freedom due to the vertical motion and the wheel has one degree of freedom corresponding to the rolling motion. The five-point suspension has one degree of freedom. The inertia characteristics of the rigid bodies (masse and inertia) are presented in Table A1 in Appendix A. It is clear that the lower link has zero mass and, in turn, zero inertia but it will recieve contribution from adjacent bodies at the connecting joints when constructing the equivalent system of particles. The characteristics of the suspension springs and dampers and the wheel are presented in Table A2 and A3, respectively.



(a) With body numbers, Joint types, and joint velocity variables



(b) Withthe primary particles and the body attached coordinate frames

Fig. 4 Schematic view of the multi-link five-point suspension

Table A1 Description of the rigid bodies

Body	Description	Mass (kg)	Inertia (kg·m^2) ξξ·ηη·ζζ·ηζ·ξξ·ζη
1	Main chassis	456.0	570.0, 2320.0, 2715.0, 0.0, 0.0, 0.0
2	Lower link	0.0	0.0, 0.0, 0.0, 0.0, 0.0, 0.0
3	Knuckle	11.8	0.5, 0.5, 0.5, 0.0, 0.0, 0.0
4	Wheel	25.0	1.0, 2.0, 1.0, 0.0, 0.0, 0.0

Table A2 The characteristics of the suspension springs and dampers

No.	Connected bodies	K (N/m)	D (N sec/m)	l_o (m)
1	(1,2)	5.11E+04	0.00E+00	0.3512
2	(1,2)	0.0E+00	1.6E+03	0.0

Table A3 The characteristics of the wheels

Radius	0.35 m
Stiffness	6.0E+06 Nm
Damping coefficient	1.0E+04 N sec/m

Each rigid body is replaced by its equivalent system of particles, as shown in Fig. 4(b). The main chassis is replaced with the four-primary particles representation (particles 1, 2, 3, and 4) and the knuckle is replaced by the four-primary particles representation (particles 6, 7, 8, and 9). The steering rod is modelled with a system of two primary particles (3 and 10) and the wheel axis is represented with particles (11 and 12). Locating the primary particles belonging to adjacent bodies together at the connection joints reduces the total number of particles replacing the whole system and leads to the automatic elimination of the kinematic constraints at these joints. The closed loop system can be reduced to open-loop system by modelling the four massless links, as shown in Fig. 4(a), as four spherical-spherical joints and then, cut these joints. Consequently, the system is reduced to a serial branch connecting the chassis to the knuckle through the lower link. An overall equivalent system of 12 particles is constructed as shown in Fig. 4(b). Cutting the four spherical-spherical joints results in cut-joint constraint equations that are expressed in terms of the Cartesian coordinates of the particles. The cut-joint constraints have the form

$$(\mathbf{r}_2-\mathbf{r}_8)^T(\mathbf{r}_2-\mathbf{r}_8)-d_{2,8}^2=0 \tag{23a}$$

$$(\mathbf{r}_3-\mathbf{r}_{10})^T(\mathbf{r}_3-\mathbf{r}_{10})-d_{3,10}^2=0 \tag{23b}$$

$$(\mathbf{r}_4-\mathbf{r}_7)^T(\mathbf{r}_4-\mathbf{r}_7)-d_{4,7}^2=0 \tag{23c}$$

$$(\mathbf{r}_5-\mathbf{r}_9)^T(\mathbf{r}_5-\mathbf{r}_9)-d_{5,9}^2=0 \tag{23d}$$

where $d_{i,j}$ is the distance between particle i and j. The equations of motion are generated recursively along the serial branch as discussed in Sec. 4 with the introduction of the cut-joint constraints.

The equations of motion are used to simulate the free response of the system from the rest position. Figure 5 and 6 presents, respectively, the time variation of the vertical displacement and acceleration of the center of the chassis. The main chassis is accelerated downward due to the gravitational forces. Then, it undergoes a damped oscillatory motion controlled by the spring-damper-actuator elements forces and the wheel compression forces up to the steady state. It should be noted that for the absolute coordinate formulation, a system of 22+15 differential equations of motion plus algebraic equations of kinemtaic constraints, respectively is constructed. Thus a resulting system of 37 differential-algebraic equations should be solved at every time step to determine the unknown accelerations and reaction forces. In the present formulation, a resulting system 27 differential-algebraic equations (5 differential equations of motion + 22 geometric constraints) is constructed. The reduction in the number of differential equations and, in turn, the number of integration variables obtained using the present formulation is considered as an advantage over the absolute coordinate formulation.

It should be pointed out that in this formulation, the kinematic constraints due to some

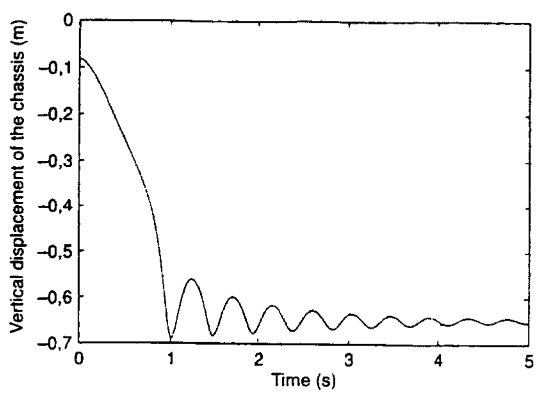


Fig. 5 The time variation of the vertical displacement of the chassis

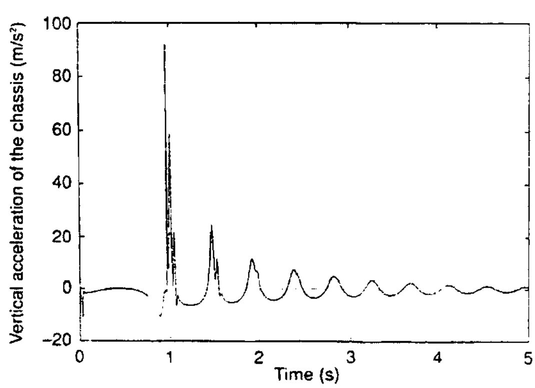


Fig. 6 The time variation of the vertical acceleration of the chassis

common types of kinematic joints (e.g. revolute or spherical joints) can be automatically eliminated by properly locating the equivalent particles. The remaining kinematic constraints along with the geometric constraints are, in general, either linear or quadratic in the Cartesian coordinates of the particles. Therefore, the coefficients of their Jacobian matrix are constants or linear in the rectangular Cartesian coordinates. Where as in the formulation based on the relative coordinates, the constraint equations are derived based on loop closure equations which have the disadvantage that they do not directly determine the positions of the links and points of interest which makes the establishment of the dynamic problem more difficult. Also, the resulting constraint equations are highly nonlinear and contain complex circular functions. The absence of these circular functions in the point coordinate formulation leads to faster convergence and better accuracy. Furthermore, preprocessing the mechanism by the topological graph theory is not necessary as it would be the case with loop constraints.

Also, in comparison with the absolute coordinates formulation, the manual work of the local axes attachment and local coordinates evaluation as well as the use of the rotational variables and the rotation matrices in the absolute coordinate formulation are not required in the point coordinate formulation. This leads to fully computerized analysis and accounts for a reduction in the computational time and memory storage. In addition to that, the constraint equations take much simpler forms as compared with the absolute coordinates.

The elimination of the rotational coordinates, angular velocities and angular accelerations in the presented formulation, leads to possible savings in computation time when this procedure is compared against the absolute or relative coordinate formulation. It has been determined that numerical computations associated with rotational transformation matrices and their corresponding coordinate transformations between reference frames is time consuming and, therefore, if these computations are avoided, more efficient codes may be developed (Attia, 1993 ; Nikravesh and Affifi, 1994). The elimination of rotational coordinates can also be found very beneficial in design sensitivity analysis of multibody systems. In most procedures for design sensitivity analysis, leading to an optimal design process, the derivatives of certain functions with respect to a set of design parameters are required. Analytical evaluation of these derivatives are much simpler if the rotational coordinates are not present and if we only deal with translational coordinates.

Some practical applications of multibody dynamics require one or more bodies in the system to be described as deformable in order to obtain a more realistic dynamic response (Attia, 1993 ; Nikravesh and Affifi, 1994). Deformable bodies are normally modeled by the finite element technique. Assume that the deformable body is connected to a rigid body described by a set of particles. Then, one or more particles of the rigid body can coincide with one or more nodes of the deformable body in order to describe the kinematic joint between the two bodies. This is a much simpler process that when the rigid body is described by a set of translational and rotational coordinates.

Also, since we are dealing in this formulation with a system of particles instead of rigid bodies, only the laws of particle dynamics are utilized in generating the equations of motion of the mechanical system. This makes the formulation much simpler than the other dynamic formulations which use the rigid body dynamical equations of motion both translational and rotational. In summary, the methodologies presented in this paper have many interesting characteristics which may be found useful in some applications. These methodologies can be combined with other methods to develop even more efficient, accurate, and flexible procedures. It should be noted that there is no single multibody formulation to be considered as the best formulation for general multibody dynamics. Each formulation has its own unique or common features and, therefore, selected features should be adopted to our advantages (Attia, 1993 ; Nikravesh and Affifi, 1994).

Since the current trend in multibody dynamics formulation is towards recursive formulations instead of matrix formulations due to its efficiency (De Jalon et al., 1994), the presented formulation has an advantage over the matrix formulation presented in (Nikravesh and Affifi, 1994). Also, the elimination of the necessity of the transformation to relative joint variables and the distribution of the external forces and couples acting on the bodies over the particles is considered as an additional advantage for the present formulation over the matrix formulation.

4. Conclusion

In the present work, a recursive formulation for the spatial motion of a system of rigid bodies is presented. The concepts of linear and angular momenta are used to formulate the rigid body dynamical equations of motion which are expressed in terms of the rectangular Cartesian coordinates of a dynamically equivalent constrained system of particles. This groups the advantages of the automatic elimination of the unknown internal constraint forces, the absence of any rotational coordinates in addition to the rotational transformation matrices, and the elimination of the necessity to distribute the external forces and force couples over the particles. Also, the formulation can be considered as a natural extension to the finite element representation for a deformable body. Some useful geometric relations are used which result in a reduced system of differential-algebraic equations. The formulation can be applied to open and/or closed-chain with the common types of kinematic joints and may be found useful in some applications. The formulation is applied to simulate the dynamic response of a multi-link five-point suspension.

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